Because Eq. (18) represents the norm of a vector, and because I is positive definite, $I^{-1/2}Gx$ itself must be zero. Therefore, x is shown to be in the null space of G. Moreover, because the null spaces of G and K are equal, the multiplicity of the zero eigenvalue is unchanged by the reformulation. This was not the case for the nonzero eigenvalues where the reformation, Eq. (13), doubled the multiplicity.

After the solution of the reformulated eigenvalue problem has been obtained, the rigid-body mode component of the eigenvector has to be reconstructed from the velocity information, as it is not explicitly present in the reduced state vector. The velocity term is written

$$\dot{\eta}=\lambda\eta \tag{19}$$

where $\lambda=i\omega$. Denoting by η_1 and η_2 the real and imaginary part of the modal vector η, respectively, the velocity becomes

$$\dot{\eta}=-\omega\eta_2+i\omega\eta_1 \tag{20}$$

Then, the real and imaginary parts of the eigenvector x_r are simply

$$y_r=\left\{\begin{matrix}-\omega\eta_2\\ \eta_{1_E}\end{matrix}\right\}_r \qquad z_r=\left\{\begin{matrix}\omega\eta_1\\ \eta_{2_E}\end{matrix}\right\}_r \tag{21}$$

In order to construct the eigenvector q_r in the configuration space, we recall Eq. (4) and write

$$q_r=\phi\eta_r=\phi\eta_{1r}+i\phi\eta_{2r} \tag{22}$$

where η_{1r} and η_{2r} are the real and imaginary parts of the eigenvector η_r.

Conclusions

A practical method is presented that allows the analyst to properly assess the effect of stored angular momentum on the system vibration properties. From a numerical standpoint, the method is attractive because it uses existing capabilities to model the structure with the rotors locked, and then avoids complex quantities by executing an eigenvalue analysis of a reformulated symmetric matrix. This method leads directly to the solution of the response problem.[2] Parametric analysis of the system for different rotor angular momentum requires the execution of the reformulated gyroscopic eigenvalue problem only. Advantage may be taken of the ability to truncate the number of input modes in order to make the parametric analysis less burdensome. The reformulated gyroscopic problem may also be run alone to study the effect of different rotor locations. This assumes that the rotors are small so that when their locations are changed, they do not greatly affect the classical modes.

Acknowledgment

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Stationkeeping at Constant Distance

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Introduction

A STATIONKEEPING procedure is described in which a vehicle circles about an orbiting body while maintaining constant separation distance. The origin of the problem was a requirement for the Orbiter of the Space Shuttle to inspect and check out a deployed payload. Most previous studies of relative motion of bodies close to one another have focused on the terminal phase of rendezvous,[1-4] while others have considered flight in formation without the requirement to view all sides of one of the vehicles.[5-7] One study which presents examples of the circling type of relative orbits[8] does not consider the problem of maintaining a constant separation distance between the two bodies, which is of essential interest in the current study.

Stationkeeping at Constant Distance

The Clohessy-Wiltshire equations[1] describe the relative motion of two bodies in orbit. These equations were examined and a solution found that results in the Orbiter making a circular orbit relative to the payload so that the separation distance between the Orbiter and payload remains constant at 3000 ft, a value chosen from safety considerations. The equations in the coordinate system employed are:

$$x=2(2\dot{x}_0/\omega-3z_0)\sin\omega t-(2\dot{z}_0/\omega)\cos\omega t + (6\omega z_0-3\dot{x}_0)t+(x_0+2\dot{z}_0/\omega)$$

$$y=y_0\cos\omega t+(\dot{y}_0/\omega)\sin\omega t$$

$$z=(2\dot{x}_0/\omega-3z_0)\cos\omega t+(\dot{z}_0/\omega)\sin\omega t+(4z_0-2\dot{x}_0/\omega)$$

where t is time, s; x is the distance along reference orbit plus downrange, ft; y is the distance normal to reference orbit, ft (makes right-hand system with x and z); z is the distance below reference orbit, ft; ω is 2π/orbit period, 1/s; and subscript 0 denotes initial condition.

Note that the motion in the crossrange (y) direction is independent of the vertical and downrange z and x motion. The amplitude of the periodic motion in the downrange direction can be seen to be twice that in the vertical direction.

For the purposes of this analysis, the following assumptions are made concerning the initial conditions.

At $t=0$, assume that 1) a phasing orbit has been completed so that the payload will be in the center of the stationkeeping orbit, $x_0=-(2\dot{z}_0/\omega)$, 2) otherwise, the Shuttle orbit is the same as the payload orbit, $y_0=z_0=0$, and 3) to prevent drift, $\dot{x}_0=0$.

Now, consider motion in the x-z plane only. This is the vertical plane aligned along the direction of motion in the parking orbit.

$$x=-(2\dot{z}_0/\omega)\cos\omega t$$

$$z=(\dot{z}_0/\omega)\sin\omega t$$

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The square of the separation distance between the payload and Orbiter can be determined as

$$r^2 = x^2 + z^2 = (\dot{z}_0/\omega)^2 (1 + 3\cos^2\omega t)$$
$$= \tfrac{1}{2}(\dot{z}_0/\omega)^2 (5 + 3\cos 2\omega t)$$

The period of variation of the separation distance can be seen to be one-half the orbit period of the parking orbit. The maximum separation distance is twice the minimum. Such motion is illustrated later.

Now add motion in the y direction also, but delay its initiation by an arbitrary time t_0.

$$y = (\dot{y}_0/\omega)\sin\omega(t - t_0)$$

The separation distance is now

$$r^2 = x^2 + y^2 + z^2$$

$$2r^2 = 5(\dot{z}_0/\omega)^2 + (\dot{y}_0/\omega)^2 - [(\dot{y}_0/\omega)^2 \sin 2\omega t_0]\sin 2\omega t$$
$$+ [3(\dot{z}_0/\omega)^2 - (\dot{y}_0/\omega)^2 \cos 2\omega t_0]\cos 2\omega t$$

The amplitude of the periodic variation in $2r^2$ is

$$\sqrt{[(\dot{y}_0/\omega)^2 \sin 2\omega t_0]^2 + [3(\dot{z}_0/\omega)^2 - (\dot{y}_0/\omega)^2 \cos 2\omega t_0]^2}$$

Values of $\omega t_0 = 0$, π, etc., correspond to minimum values of the amplitude, while $\omega t_0 = \pi/2$, $3\pi/2$, etc., give maximum values.

Consider first the minimum amplitude situation for $t_0 = 0$. The minimum and maximum values of the separation distance from the payload to the Orbiter become

$$r_{\text{min.}} = 2(\dot{z}_0/\omega) \qquad r_{\text{max.}} = \sqrt{(\dot{z}_0/\omega)^2 + (\dot{y}_0/\omega)^2}$$

In order to obtain a constant separation distance, these are equated. This relates the initial velocities required in the vertical and horizontal directions (and the resulting amplitudes of motion).

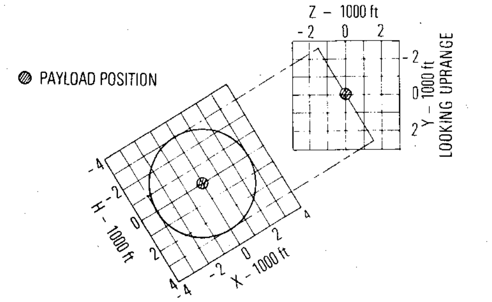
$$\dot{y}_0 = \pm\sqrt{3}\,\dot{z}_0$$

The resultant of the two velocities makes a 30-deg angle to the horizontal, and the orbit of the Orbiter relative to the payload will be inclined similarly. The existence of the plus or minus sign in the solution indicates that the orbit may be inclined either to the right or the left relative to the horizontal.

The initial velocities required for a 3000-ft separation for a 172-naut mile circular orbit can be determined as:

$\Delta V_{\text{TOT}} = \sqrt{(\dot{y}_0)^2 + (\dot{z}_0)^2} = 2\dot{z}_0$
$r_{\text{min.}} = 3000$ ft
$\omega = 1.154 \times 10^{-3}$ 1/s for a 172-naut mile circular orbit
$\dot{z}_0 = r_{\text{min.}}\omega/2 = 1.73$ fps
$\dot{y}_0 = 3.00$ fps
$\Delta V_{\text{TOT}} = 3.46$ fps

The procedure for establishing this orbit involves a phasing orbit to position the Orbiter so that the payload will be in the center of the stationkeeping orbit, as assumed in the initial conditions for the analysis. Initially, the Orbiter and its payload are side by side in a 172-naut mile circular orbit. A retrovelocity ΔV_l of 0.18 fps is applied to the Orbiter, which puts it into a 171.895 × 172 naut mile orbit. This is an orbit of a slightly shorter period than that of the payload and, consequently, the Orbiter is 3000 ft ahead after one orbit revolution. At this time, a second velocity impulse of 3.47 fps is applied. This is the resultant of the 0.18 fps in the downrange direction to restore the Orbiter to its original orbital velocity, 1.73 fps upward to initiate vertical and downrange motion, and 3.00 fps in the horizontal plane to initiate crossrange motion.



Note: A relative orbit plane sloping downward to the right is also satisfactory as well as the orientation shown which slopes downward to the left

Fig. 1 Stationkeeping at constant distance – relative orbit.

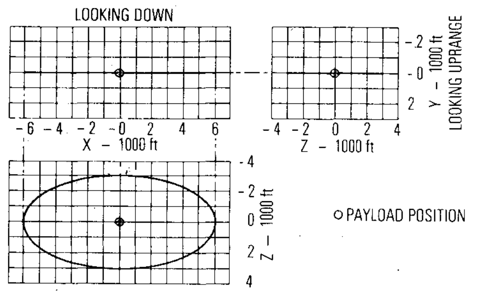


Fig. 2 Stationkeeping in vertical plane – relative orbit.

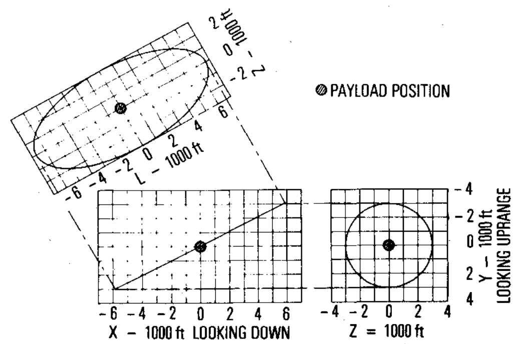


Fig. 3 Stationkeeping with vertical and horizontal motion – phased for maximum difference between minimum and maximum separation – relative orbit.

Computer simulations[9,10] were made of this stationkeeping mode over a spherical Earth with a parking orbit inclination of 98.7 deg. The fully established motion, as determined by the trajectory simulation, is illustrated in Fig. 1. An edge-on view of the motion looking uprange is presented, which shows the motion to be in a plane inclined at 30 deg to the horizontal. (A plane inclined at −30 deg would also be satisfactory.) The edge of the plane is aligned with the direction of motion in the parking orbit. The other view shows the relative motion to be in a 3000-ft radius circle with the payload at the center.

The same procedure was repeated for an oblate Earth; no significant change in the relative motion was evident. Hence, all further simulations were made assuming a spherical Earth.

Other Stationkeeping Orbits

Some of the consequences of deviating from the stationkeeping procedure yielding constant distance are presented. Motion in the vertical plane only with a minimum separation distance of 3000 ft is shown in Fig. 2. The horizontal motion is twice the vertical motion. The total separation varies from 3000 to 6000 ft at a period equal to half the period of the parking orbit.

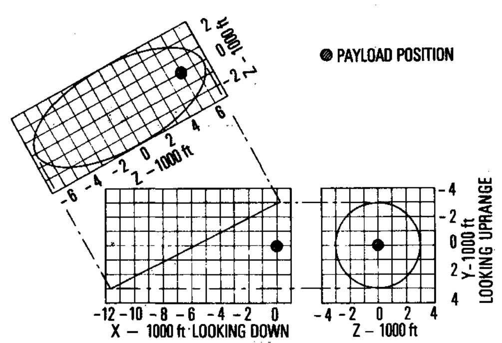


Fig. 4 Stationkeeping for worst phasing of vertical and horizontal motion and no initial phasing in downrange – relative orbit.

The other set of values for t_0, $\omega t_0 = \pi/2$, $3\pi/2$, etc., will now be considered. These values yield the maximum difference between the minimum and maximum separation. In this case,

$$r_{\text{min.}} = \dot{z}_0/\omega \qquad r_{\text{max.}} = \sqrt{4(\dot{z}_0/\omega)^2 + (\dot{y}_0/\omega)^2}$$

The difference between $r_{\text{min.}}$ and $r_{\text{max.}}$ can be minimized by eliminating the motion in the y direction completely ($\dot{y}_0 = 0$), in which case the maximum separation is twice the minimum value. This is just the motion in the vertical plane only.

Crossrange motion with an amplitude of 3000 ft will now be added. The phase of the crossrange motion relative to the vertical motion corresponds to that for maximum difference between minimum and maximum separation ($\omega t_0 = \pi/2$). This is illustrated in Fig. 3. Looking uprange, the relative orbit appears to be circular, although in true projection the orbit is elliptical. The increased range of separation distance is clearly evident.

Finally, if the initial phasing maneuver is eliminated from the stationkeeping initiation procedure, the motion along the flight path is entirely in the uprange direction, as shown in Fig. 4. This results in substantial increases in the maximum separation compared to the minimum. In addition, it results in the introduction of a component in the total separation distance with a period equal to the parking orbit period in addition to the half parking orbit period found previously.

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Choice of Method for Discretization of Continuous Systems

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Introduction

WHEN a digital computer is used to replace a continuous system for signal processing or control, it is necessary to convert the continuous process mathematical description into a discrete mathematical model, usually with difference equations. For example, digital filtering and simulation usually make this conversion necessary. The specific motivation for our work was digital autopilots, which may be classified as a real-time simulation application in that the goal was to replace existing hardware with a computer and software and then use a D/A (digital-to-analog) device to convert the numerical data back to a signal for aircraft control. The D/A converter, of course, is modeled as a zero-order hold.

A diagram which contains the essential functions is shown in Fig. 1. Input signals are sampled and converted to numbers; the linear transfer function to be implemented is described in a discrete manner with a method such as the z transform or Tustin or a numerical method such as Runge-Kutta. A data reconstruction device converts the computer output back into a continuous signal to operate the system. Only those situations described by Fig. 1 are of concern here, because the cases where output is used in the form of information have been the primary concern of the literature of discrete signal processing. And, judging from the literature, situations in which a signal output is required have received little attention. Perhaps this is because only recently has the revolution in small inexpensive computers placed an emphasis on systems working in real time whose outputs are converted to continuous signals.

In the case of an autopilot one could start over with new specifications and design a digital autopilot, but for existing systems it might be desired to simply discretize the existing transfer functions and make them function as nearly like the continuous system as needed for acceptance and satisfactory performance. This is the case of most digital filters, too, because of the rich body of classical filter design information that is available. Thus, the discretization problem where processed signals must be returned to the system is unavoidable; and it is easy to prove that, in general, it is not possible to find exact discrete equivalents to continuous systems. The principle differences of the output signals of a continuous processor and its digital replacement will be 1) phase shift and amplitude changes due to the hold device at

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